HW ~~4~~5 Solutions

1.) $-\Delta(u-\Phi) \le 0$

$\Rightarrow \max_{\bar{U}} u-\Phi \le \max_{\partial U} u-\Phi = 0$

So $u-\Phi \le 0$ on U

or $u \le \Phi$ on U

2.) Part A.) This is just the Mean Value Formula/
Corollary 9.4

Part B.) $B(x_0/2, \beta) \subseteq B(x_0, R)$ $\quad \beta = R - r_0/2$

gives $\operatorname{Vol}(B(0,R) \setminus B(x_0,R)) \le \operatorname{Vol}(B(0,R) \setminus B(x_0/2, \beta))$

and $\operatorname{Vol}(B(0,R) \setminus B(x_0/2, \beta)) =$

$\operatorname{Vol}(B(0,R)) - \operatorname{Vol}(B(x_0/2, \beta))$

$= \frac{A_n}{n}[R^n] - \frac{A_n}{n}(R - r_0/2)^n$

c.) $|u(0) - u(x_0)| \le \frac{n}{A_n R^n}\left[\left|\int_{B(0,R)} u\,dx - \int_{B(x_0,R)} u\,dx\right|\right]$

$\le \frac{n}{A_n R^n}\left[\int_{B(0,R)\setminus B(x_0,R)} |u|\,dx + \int_{B(x_0,R)\setminus B(0,R)} |u|\,dx\right]$

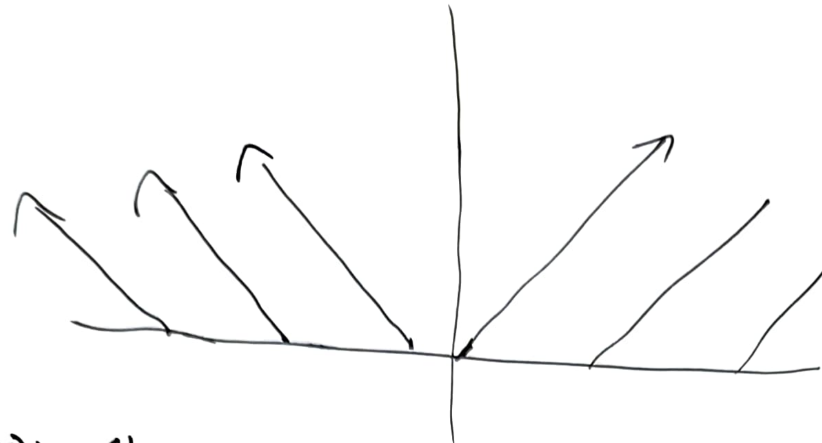
$\le \frac{n}{A_n R^n}\left[2M \cdot \frac{A_n}{n}\left[R^n - (R - r_0/2)^n\right]\right]$

$\le 2M\left[1 - \frac{(R - r_0/2)^n}{R^n}\right]$

as $R \to \infty$, this $\to 0$

So $|u(0) - u(x_0)| = 0$

3.) A.)



$\rightsquigarrow$ no Characteristics touch $x=0$

B.) $G'(t) = 1-b-a \qquad G(0)=0$

$\iff G(t) = 0$

C.) $G_1(t) = -\frac{1}{2}t \qquad G_2(t) = \frac{1}{2}t$

D.) $\frac{\partial}{\partial t}\left(\frac{1}{2}-\frac{x}{2t}\right) + \left(1-2\left(\frac{1}{2}-\frac{x}{2t}\right)\right)\frac{\partial}{\partial x}\left(\frac{1}{2}-\frac{x}{2t}\right)$

$= \frac{x}{2t^2} + 2\cdot\frac{x}{2t}\left(-\frac{1}{2t}\right) = 0$

$G_1(t) = -t$ has $G_1'(t) = -1$

and at $x = -t$

$\frac{q(u_+)-q(u_-)}{u_+ - u_-} = 1 - u_+ - u_- = 1-1-1 = -1$

Similarly, at $G_2(t) = t$, $G_2'(t) = 1$

$\frac{q(u_+)-q(u_-)}{u_+ - u_-} = 1-0-0 = 1$

4.) A.) $$\int_{\mathbb{R}} \left(x \frac{\partial u}{\partial x} - 1\right)\varphi \, dx = 0$$

$$\iff \int_{\mathbb{R}} -u \frac{\partial}{\partial x}(x\varphi) - \varphi \, dx = 0$$

$\rightsquigarrow$ only need $u \in L^1_{loc}$

B.) $$\int_{\mathbb{R}} -\log|x|(x\varphi' + \varphi) - \varphi \, dx$$

$$= \int_{\mathbb{R}} \varphi'(-x\log|x|) + \varphi(-1-\log|x|)\,dx$$

$$= \int_{\mathbb{R}} -\varphi(-\log|x| - 1) + \varphi(-1-\log|x|)\,dx$$

$$= 0.$$

as desired.